

A HOCHSCHILD'S 6-TERM EXACT SEQUENCE FOR RESTRICTED LIE SUPERALGEBRAS

GONGXIANG LIU

ABSTRACT. In [5], Hochschild established a 6-term exact sequence for the cohomology of restricted Lie algebras. We generalize this result to restricted Lie superalgebras.

1. INTRODUCTION

1.1. As generalizations and deep continuations of classical Lie theory, Lie superalgebras over the field of complex numbers \mathbb{C} have been studied extensively since the classification of finite dimensional complex simple Lie superalgebras by Kac [7]. Comparing with abundant works and results for various cohomology theory of Lie superalgebras (see [1, 6, 8, 10] and references therein), the knowledge about the cohomology theory of restricted Lie superalgebras is poor. To the author's best knowledge, there has not been any serious study in the direction, perhaps because even for simple Lie superalgebras over \mathbb{C} their cohomology theory is already very difficult.

1.2. In [5], Hochschild gave a pioneering trial to the cohomology theory for restricted Lie algebras and as a final conclusion a 6-term exact sequence was obtained:

$$\begin{aligned} 0 \rightarrow H_*^1(L, M) &\rightarrow H^1(L, M) \rightarrow S(L, M^L) \\ &\rightarrow H_*^2(L, M) \rightarrow H^2(L, M) \rightarrow S(L, H^1(L, M)), \end{aligned}$$

for L a restricted Lie algebra and M a strongly abelian restricted Lie algebra with an L -operation. Here H^i and H_*^i denote the “ordinary” situation's cohomology groups and “restricted” ones respectively, $S(V, W)$ is the space of p -semilinear maps from V to W , and M^L is the subset of invariants, i.e., $M^L = \{m \in M \mid L \cdot m = 0\}$. This 6-term exact sequence established the connection between ordinary cohomology groups and restricted ones, and was shown to be crucial to get further information about cohomology theory of restricted Lie algebras, algebraic groups, infinitesimal groups and discrete groups [2, 3]. Especially, it can help us to establish Noetherian property for cohomology algebra $H^*(\mathfrak{g}, k)$ for a restricted Lie algebra \mathfrak{g} . The current work

generalizes this 6-term exact sequence to restricted Lie superalgebras

$$\begin{aligned} 0 \rightarrow H_*^1(\mathfrak{g}, M) &\rightarrow H^1(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, M_0^{\mathfrak{g}}) \\ &\rightarrow H_*^2(\mathfrak{g}, M) \rightarrow H^2(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, H^1(\mathfrak{g}, M)). \end{aligned}$$

See Theorem 5.7 for details. We hope that the result we gotten can be used as an experimental animal to detect whether the cohomology algebra of a Lie superalgebra is finitely generated or not.

1.3. In Section 2, some necessary notions and results are collected. In particular, we show that the ordinary (co)homology of a Lie superalgebra defined by its Koszul complex can be computed through the Hochschild complex of its enveloping algebra. As we expect, the extensions of restricted supermodules can be explained through the first cohomology group H_*^1 . The proof of this fact is given in Section 3. We give the cohomological interpretations to the similarity classes and equivalence classes of extensions of restricted Lie superalgebras in Section 4,5 respectively. And as a result, we get the desired 6-term exact sequence. The results gotten in the paper are what one would naturally hope them to be for restricted Lie superalgebras.

Throughout we work with a field k with characteristic $p > 2$ as the ground field. By a superspace we mean a \mathbb{Z}_2 -graded vector space $V = V_0 \oplus V_1$, in which we call elements in V_0 and V_1 even and odd, respectively. Write $|v| \in \mathbb{Z}_2$ for the degree of $v \in V$, which is implicitly assumed to be \mathbb{Z}_2 -homogeneous. A linear map $f : V = V_0 \oplus V_1 \rightarrow W = W_0 \oplus W_1$ is said to be even (resp. odd) if $f(V_i) \subseteq W_i$ (resp. $f(V_i) \subseteq W_{i+1}$) for $i = 0, 1$. Unless otherwise specified, all vector spaces, algebras, subalgebras, ideals, modules and submodules etc. are in the super case, and all linear maps are even. Moreover, for any two \mathbb{Z}_2 -graded vector spaces V, W , we use $\text{Hom}_k(V, W)$ to represent the set of all even linear maps from V to W and $\underline{\text{Hom}}_k(V, W)$ to denote that of all linear maps. A map f from V to W is p -semilinear if $f(\alpha v_1 + v_2) = \alpha^p f(v_1) + f(v_2)$ for $\alpha \in k$ and $v_1, v_2 \in V$. And, we use the notation $S(V, W)$ to denote the space of p -semilinear maps from V to W .

2. BASIC RESULTS FOR THE COHOMOLOGY OF (RESTRICTED) LIE SUPERALGEBRAS

The materials in this section are standard generalization from Lie algebras to Lie superalgebras except Lemma 2.2, where we need give a generalization of the sign representation of a symmetric group.

2.1. Basic notions. The definition of a restricted Lie superalgebra can be easily formulated (cf. e.g. [11]).

Definition 2.1. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called a *restricted Lie superalgebra*, if there is a p th map $\mathfrak{g}_{\bar{0}} \rightarrow \mathfrak{g}_{\bar{0}}$, denoted as $^{[p]}$, satisfying

- (a) $(cx)^{[p]} = c^p x^{[p]}$ for all $c \in k$ and $x \in \mathfrak{g}_{\bar{0}}$,
- (b) $[x^{[p]}, y] = (adx)^p(y)$ for all $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}$,
- (c) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ for all $x, y \in \mathfrak{g}_{\bar{0}}$ where is_i is the coefficient of λ^{i-1} in $(ad(\lambda x + y))^{p-1}(x)$.

In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module by the adjoint action of the even subalgebra. For a Lie superalgebra \mathfrak{g} , $U(\mathfrak{g})$ is denoted to be its universal enveloping algebra and $\mathbf{u}(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} | x \in \mathfrak{g}_{\bar{0}})$ its restricted enveloping algebra if moreover \mathfrak{g} is restricted.

The notion of the cohomology for a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ was introduced by Fuks (cf. e.g. [4]). By definition, the (ordinary) space of n -dimensional cocycles of \mathfrak{g} with coefficients in the \mathfrak{g} -module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is defined to be

$$C^n(\mathfrak{g}, M) := \bigoplus_{n_0+n_1=n} \text{Hom}_k(\Lambda^{n_0} \mathfrak{g}_{\bar{0}} \otimes S^{n_1} \mathfrak{g}_{\bar{1}}, M).$$

The differential $\delta_n : C^n(\mathfrak{g}, M) \rightarrow C^{n+1}(\mathfrak{g}, M)$ is defined by the formula

$$\begin{aligned} \delta_{n-1}(f)(x_1, \dots, x_{n_0}, y_1, \dots, y_{n_1}) \\ = & \sum_{s=1}^{n_0} (-1)^{s-1} x_s \cdot f(x_1, \dots, \hat{x}_s, \dots, x_{n_0}, y_1, \dots, y_{n_1}) \\ & + \sum_{t=1}^{n_1} (-1)^{n_0} y_t \cdot f(x_1, \dots, x_{n_0}, y_1, \dots, \hat{y}_t, \dots, y_{n_1}) \\ & + \sum_{1 \leq s < t \leq n_0} (-1)^{s+t} f([x_s, x_t], x_1, \dots, \hat{x}_s, \dots, \hat{x}_t, \dots, x_{n_0}, y_1, \dots, y_{n_1}) \\ & + \sum_{s=1}^{n_0} \sum_{t=1}^{n_1} (-1)^s f(x_1, \dots, \hat{x}_s, \dots, x_{n_0}, [x_s, y_t], y_1, \dots, \hat{y}_t, \dots, y_{n_1}) \\ & + \sum_{1 \leq s < t \leq n_1} -f([y_s, y_t], x_1, \dots, x_{n_0}, y_1, \dots, \hat{y}_s, \dots, \hat{y}_t, \dots, y_{n_1}). \end{aligned}$$

For $x_1, \dots, x_n \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$, one can give a more unified expression for the differential

$$\begin{aligned} \delta_{n-1}(f)(x_1, \dots, x_n) \\ = & \sum_{s=1}^n (-1)^{s-1+|x_s|(\sum_{i=1}^{s-1} |x_i|)} x_s \cdot f(x_1, \dots, \hat{x}_s, \dots, x_n) \\ & + \sum_{1 \leq s < t \leq n_0} (-1)^{s+t+|x_s|(\sum_{i=1}^{s-1} |x_i|)+|x_t|(\sum_{i=1}^{t-1} |x_i|)+|x_s||x_t|} \\ & \cdot f([x_s, x_t], x_1, \dots, \hat{x}_s, \dots, \hat{x}_t, \dots, x_n). \end{aligned}$$

It is straightforward to show that $\delta_{n+1} \circ \delta_n = 0$ and hence, in particular, one can define the cohomologies by setting

$$H^n(\mathfrak{g}, M) := \text{Ker } \delta_n / \text{Im } \delta_{n-1}.$$

We call it the n -th cohomology of \mathfrak{g} with coefficients in M .

Also one can use the usual Hochschild's complex to define the cohomologies for any augmented algebra. In our case, let $U(\mathfrak{g})^+$ be the ideal in $U(\mathfrak{g})$ generated by \mathfrak{g} . The n -cochains are now the *even* n -linear functions on $U(\mathfrak{g})^+$ with values in M , and the coboundary operator δ is defined by the formula

$$\delta_{n-1}(f)(s_1, \dots, s_n) = s_0 \cdot f(s_2, \dots, s_n) + \sum_{i=1}^{n-1} (-1)^i f(s_1, \dots, s_i s_{i+1}, \dots, s_n),$$

for $s_1, \dots, s_n \in U(\mathfrak{g})^+$.

At the first glance, it is not so clear whether the cohomologies as defined above in the two different ways are same or not. For convenience, we call the cochains defined in the first way and the second way, the *Lie* type and *associative* type, respectively.

Lemma 2.2. *There is a canonical isomorphism between the cohomology groups of Lie type and associative type.*

Proof. We give an explicit cochain map between two complexes defined as above. To do it, we need introduce a notation at first. Let S_n be the symmetric group in n letters. For any $\sigma \in S_n$ and $1 \leq n_0 \leq n$, define

$$\text{sgn}(\sigma(n_0|n)) := (-1)^{\sigma(\bar{1}) + \dots + \sigma(\bar{n})}$$

where $\sigma(\bar{i}) := \#\{j \in \{1, \dots, n_0\} | j \notin \{\sigma(1), \dots, \sigma(i-1)\}, j < \sigma(i)\}$ for $1 \leq i \leq n$. Now, for every cochain f of associative type, define a cochain f' of Lie type by the formula

$$(2.1) \quad f'(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma(n_0|n)) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where we assume that $x_1, \dots, x_{n_0} \in \mathfrak{g}_0$ while $x_{n_0+1}, \dots, x_n \in \mathfrak{g}_1$. One can verify directly the $\delta(f') = (\delta(f))'$ and indeed the map $f \mapsto f'$ induces an isomorphism of the cohomology groups. \square

Remark 2.3. The notion $\text{sgn}(\sigma(n_0|n))$ generalizes our usual sign representation $\text{sgn} : S_n \rightarrow \{\pm 1\}$. In fact, we always have

$$\text{sgn}(\sigma(n|n)) = \text{sgn}(\sigma).$$

Therefore, the formula (2.1) generalizes the isomorphism given by Hochschild ([5], p. 557) for Lie algebra to Lie superalgebra.

Note that we can not use Lie type cochains to define the cohomology groups for a restricted Lie superalgebra directly. Compare with Lie type cochains, we still can use associative type cochains to define cohomologies for restricted Lie superalgebras. In this case, we just need replace $U(\mathfrak{g})^+$ by $\mathbf{u}(\mathfrak{g})^+$ and M by a restricted \mathfrak{g} -module. Similar to the definition of $U(\mathfrak{g})^+$, $\mathbf{u}(\mathfrak{g})^+$ is the ideal in $\mathbf{u}(\mathfrak{g})$ generated by \mathfrak{g} . In order to not cause confusion, the restricted cohomology groups are denoted by

$$H_*^n(\mathfrak{g}, M), \quad n \in \mathbb{N}.$$

Since the canonical homomorphism $U(\mathfrak{g}) \rightarrow \mathbf{u}(\mathfrak{g})$ allows us to regard any $\mathbf{u}(\mathfrak{g})$ -module also as a $U(\mathfrak{g})$ -module, there is a canonical homomorphism

$$H_*^n(\mathfrak{g}, M) \rightarrow H^n(\mathfrak{g}, M), \quad n \in \mathbb{N}.$$

An explicit cochain map inducing this homomorphism is given, in the associative type, by $f \mapsto f^0$, where $f^0(x_1, \dots, x_n) = f(x'_1, \dots, x'_n)$ with $x_i \in U(\mathfrak{g})^+$ and x'_i its canonical image in $\mathbf{u}(\mathfrak{g})^+$.

Denote the complex of the cochains for $U(\mathfrak{g})^+$ in the restricted \mathfrak{g} -module M by $C(M)$, and let $C^0(M)$ stand for the subcomplex consisting of the cochains of the form f^0 with f a cochain for $U(\mathfrak{g})^+$ in M . Then we have an exact sequence of complexes $0 \rightarrow C^0(M) \rightarrow C(M) \rightarrow C(M)/C^0(M) \rightarrow 0$ and so we get a long exact sequence

$$\cdots \rightarrow H_*^n(\mathfrak{g}, M) \rightarrow H^n(\mathfrak{g}, M) \rightarrow H^n(C(M)/C^0(M)) \rightarrow H_*^{n+1}(\mathfrak{g}, M) \rightarrow \cdots.$$

Obviously, $H^0(C(M)/C^0(M)) = 0$ and so there is an injection

$$(2.2) \quad i_1 : H_*^1(\mathfrak{g}, M) \hookrightarrow H^1(\mathfrak{g}, M).$$

2.2. Extensions. Let \mathfrak{g} be a Lie superalgebra and K, N two \mathfrak{g} -modules. An *extension of K by N* is a pair (E, ϕ) , where E is a \mathfrak{g} -module containing K , and ϕ is a \mathfrak{g} -epimorphism $E \rightarrow N$ such that $\text{Ker } \phi = K$. That is, there is an exact sequence of \mathfrak{g} -modules

$$0 \rightarrow K \rightarrow E \xrightarrow{\phi} N \rightarrow 0.$$

Two such extensions (E, ϕ) and (E', ϕ') are said to be *equivalent* if there is a \mathfrak{g} -isomorphism $\alpha : E \rightarrow E'$ which leaves the elements of K fixed and satisfies the relation $\phi'\alpha = \phi$. As usual, denote the equivalence classes of the extensions of K by N by $\text{Ext}(K, N)$, and there is an ordinary (i.e., not super) linear space structure over $\text{Ext}(K, N)$. Define M to be the k -space consisting of *all* k -linear maps from N to K , that is, using our notion

$$M = \underline{\text{Hom}}_k(N, K).$$

Through setting $M_{\bar{0}} := \underline{\text{Hom}}_k(N_{\bar{0}}, K_{\bar{0}}) \oplus \underline{\text{Hom}}_k(N_{\bar{1}}, K_{\bar{1}})$, $M_{\bar{1}} := \underline{\text{Hom}}_k(N_{\bar{0}}, K_{\bar{1}}) \oplus \underline{\text{Hom}}_k(N_{\bar{1}}, K_{\bar{0}})$, $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is a superspace. And it is indeed a \mathfrak{g} -module through

$$(2.3) \quad (x \cdot m)(a) := x \cdot m(a) - (-1)^{|x||m|} m(x \cdot a)$$

for $x \in \mathfrak{g}, m \in M$ and $a \in N$.

Lemma 2.4. *With notions as above, there is an isomorphism of k -spaces*

$$\text{Ext}(K, N) \cong H^1(\mathfrak{g}, M).$$

Proof. For any extension (E, ϕ) of K by N , let $\varphi : N \rightarrow E$ be a linear map such that $\phi\varphi = \text{id}_N$. Note that φ can be chosen as an even linear map due to ϕ is already an even homomorphism. From this, we obtain a linear map $f : \mathfrak{g} \rightarrow M$ by setting

$$f(x)(a) := x \cdot \varphi(a) - \varphi(x \cdot a)$$

for $x \in \mathfrak{g}, a \in N$. It is straightforward to show that f is a 1-cocycle and above procedure induces a linear map $F : \text{Ext}(K, N) \rightarrow H^1(\mathfrak{g}, M)$.

Conversely, for any 1-cocycle $f \in \text{Hom}_k(\mathfrak{g}, M)$. We can attach it with an extension (E_f, ϕ_f) of K by N . By definition, as vector space $E_f = K \oplus N$, $\phi_f(c, d) = d$ and the \mathfrak{g} -module is given through

$$x \cdot (c + d) := x \cdot c + x \cdot d + f(x)(d)$$

for $x \in \mathfrak{g}, c \in K$ and $d \in N$. Also, one can show this process gives a linear map $G : H^1(\mathfrak{g}, M) \rightarrow \text{Ext}(K, N)$. At last, it is direct to prove that $FG = \text{id}_{H^1(\mathfrak{g}, M)}$ and $GF = \text{id}_{\text{Ext}(K, N)}$. \square

Just like the Lie algebra case, we also hope that we can give an interpretation to the second cohomology group by using the extensions of Lie superalgebras. To attack it, let M an abelian Lie superalgebra, \mathfrak{g} an arbitrary Lie superalgebra. An *extension of M by \mathfrak{g}* is a pair (E, ϕ) , where E is a Lie superalgebra containing M as an ideal, and ϕ is a Lie superalgebra epimorphism $E \rightarrow \mathfrak{g}$ such that $\text{Ker } \phi = K$. That is, there is an exact sequence of Lie superalgebras

$$0 \rightarrow M \rightarrow E \xrightarrow{\phi} \mathfrak{g} \rightarrow 0.$$

This situation defines on M the structure of a \mathfrak{g} -module, with \mathfrak{g} operating on M via E , in the natural fashion. Similarly, two such extensions (E, ϕ) and (E', ϕ') are said to be *equivalent* if there is a Lie superalgebra isomorphism $\alpha : E \rightarrow E'$ (α is even by definition) which leaves the elements of M fixed and satisfies the relation $\phi'\alpha = \phi$. Denote the equivalence classes of M by \mathfrak{g} by $\text{Ext}(M, \mathfrak{g})$ and it is also an ordinary linear space.

Lemma 2.5. *With notions as above, there is an isomorphism of linear spaces*

$$\text{Ext}(M, \mathfrak{g}) \cong H^2(\mathfrak{g}, M).$$

Proof. Similar to the proof of Lemma 2.4, we just give the formula for the correspondence and leave the reader to check the details. For any extension (E, ϕ) of M by \mathfrak{g} , let $\varphi : M \rightarrow E$ be a linear map such that $\phi\varphi = \text{id}_M$. From this, we obtain a linear map $f \in \text{Hom}_k(\mathfrak{g} \otimes \mathfrak{g}, M)$ by setting

$$f(x_1, x_2) := [\varphi(x_1), \varphi(x_2)] - \varphi([x_1, x_2])$$

for $x_1, x_2 \in \mathfrak{g}$. It is straightforward to show that f is a 2-cocycle and above procedure induces a linear map $F : \text{Ext}(M, \mathfrak{g}) \rightarrow H^2(\mathfrak{g}, M)$.

Conversely, for any 2-cocycle $f \in \text{Hom}_k(\mathfrak{g} \otimes \mathfrak{g}, M)$. We can attach it with an extension (E_f, ϕ_f) of M by \mathfrak{g} . By definition, as vector space $E_f = \mathfrak{g} \oplus M$, $\phi_f(x_1, m_1) = x_1$ and the Lie superalgebra structure is given through

$$[(x_1, m_1), (x_2, m_2)] := ([x_1, x_2], x_1 \cdot m_2 - (-1)^{|x_1||x_2|} x_2 \cdot m_1 + f(x_1, x_2))$$

for $x_1, x_2 \in \mathfrak{g}$ and $m_1, m_2 \in M$. Here we implicitly ask (x_i, m_i) to be an homogeneous element and thus we always have $|x_i| = |m_i|$. Also, one can show this process gives a linear map $G : H^2(\mathfrak{g}, M) \rightarrow \text{Ext}(M, \mathfrak{g})$. At last, it is direct to prove that $FG = \text{id}_{H^2(\mathfrak{g}, M)}$ and $GF = \text{id}_{\text{Ext}(M, \mathfrak{g})}$. \square

Remark 2.6. Both Lemma 2.4 and Lemma 2.5 should be known by experts. The author just has not found a suitable reference.

For latter use and completeness, we collect some identities, which already appeared in [5].

Lemma 2.7. *Let $k\{x, y\}$ be the free algebra generated by two variables x, y . If D_w denote the map $z \mapsto wz - zw = D_w(z)$, then we have*

- (1) $\sum_{i=0}^{p-1} x^i y x^{p-1-i} = D_{x^{p-1}}(y)$.
- (2) $\sum_{i=0}^{l-1} x^i D_x^{l-1-i}(y) = \sum_{j=0}^{l-1} (-1)^j \binom{l}{j+1} x^{l-1-j} y x^j$.

Proof. (1) It is not hard to see that $D_x(\sum_{i=0}^{p-1} x^i y x^{p-1-i} - D_{x^{p-1}}(y)) = 0$ which implies $\sum_{i=0}^{p-1} x^i y x^{p-1-i} = D_{x^{p-1}}(y)$.

(2) Consider the commutative polynomial ring $k[x_1, x_2]$ at first. In such a ring, we always have

$$x_1^l - (x_1 - x_2)^l = x_2 \sum_{i=0}^{l-1} x_1^i (x_1 - x_2)^{l-1-i}$$

which implies

$$\sum_{i=0}^{l-1} x_1^i (x_1 - x_2)^{l-1-i} = \sum_{j=0}^{l-1} (-1)^j \binom{l}{j+1} x_1^{l-1-j} x_2^j.$$

By specializing this to our case where x_1 is the left multiplication by x in $k\{x, y\}$ and x_2 the right multiplication by x in $k\{x, y\}$, we get the desired equation. \square

3. EXTENSIONS OF RESTRICTED MODULES

In Subsection 2.2, we have considered the extensions of supermodules. Now let us consider the analogous situation in the case where \mathfrak{g} is a restricted Lie superalgebra, and K, N are restricted \mathfrak{g} -modules. Correspondingly, an extension (E, ϕ) of K by N is then called a *restricted extension* if E is a restricted \mathfrak{g} -module. Through equation (2.3), $M = \underline{\text{Hom}}_k(N, K)$ is a \mathfrak{g} -module.

Lemma 3.1. *M is also a restricted \mathfrak{g} -module.*

Proof. To show it, for any $x \in \mathfrak{g}_0$ we define two maps $u_x, v_x \in \text{Hom}_k(M, M)$. By definition, $u_x(f)(a) := x \cdot f(a)$, $v_x(f)(a) := f(x \cdot a)$ for $f \in M$ and $a \in N$. Clearly, $u_x v_x = v_x u_x$ and $x \cdot f = u_x(f) - v_x(f)$. So,

$$\begin{aligned} (x^p \cdot f)(a) &= (u_x - v_x)^p(f)(a) = (u_x^p - v_x^p)(f)(a) = x^p \cdot f(a) - f(x^p \cdot a) \\ &= x^{[p]} \cdot f(a) - f(x^{[p]} \cdot a) = (x^{[p]} \cdot f)(a) \end{aligned}$$

for $x \in \mathfrak{g}_0, f \in M$ and $a \in N$. \square

The equivalence classes of restricted extensions of K by N is denoted by $\text{Ext}_*(K, N)$. Since any restricted extension can be regarded as an ordinary extension naturally, there is a natural linear map $i_2 : \text{Ext}_*(K, N) \hookrightarrow \text{Ext}(K, N)$. The main result of this section is the following conclusion.

Proposition 3.2. *Let \mathfrak{g} be a restricted Lie superalgebra, K, N two restricted \mathfrak{g} -modules, and $M = \underline{\text{Hom}}_k(N, K)$. Then there is a canonical isomorphism $F|_* : \text{Ext}_*(K, N) \xrightarrow{\cong} H_*^1(\mathfrak{g}, M)$ such that the following diagram of canonical maps is commutative*

$$\begin{array}{ccc} \text{Ext}_*(K, N) & \xrightarrow{F|_*} & H_*^1(\mathfrak{g}, M) \\ \downarrow i_2 & & \downarrow i_1 \\ \text{Ext}(K, N) & \xrightarrow{F} & H^1(\mathfrak{g}, M) \end{array}$$

where F is the isomorphism given in Lemma 2.4 and i_1 is the injection described in (2.2).

Proof. To attack it, it is enough to show that $F i_2(\text{Ext}_*(K, N)) = \text{Im } i_1$. Actually, we will show that both $F i_2(\text{Ext}_*(K, N))$ and $\text{Im } i_1$ equal to the subspace V of $H^1(\mathfrak{g}, M)$ whose elements are represented by Lie type 1-cocycles

f satisfying

$$x^{p-1} \cdot f(x) = f(x^{[p]})$$

for $x \in \mathfrak{g}_0$.

Let (E, ϕ) be a restricted extension of K by N . By regarding it as a usual extension, we get a 1-cocycle $f \in \text{Hom}_k(\mathfrak{g}, \underline{\text{Hom}}_k(N, K))$. By definition, $f(x) = x \cdot \varphi$ for a linear section φ of ϕ and $x \in \mathfrak{g}$. Thus $x^{p-1} \cdot f(x) = x^p \cdot \varphi$. Since E and N are restricted modules, it follows that $x^{p-1} \cdot f(x) = f(x^{[p]})$ for all $x \in \mathfrak{g}_0$. Therefore, $Fi_2(\text{Ext}_*(K, N)) \subseteq V$. Conversely, let $f \in \text{Hom}_k(\mathfrak{g}, \underline{\text{Hom}}_k(N, K))$ be any Lie type 1-cocycle satisfying $x^{p-1} \cdot f(x) = f(x^{[p]})$, for $x \in \mathfrak{g}_0$. By the construction introduced in the proof of Lemma 2.4, the corresponding usual extension is denoted by (E_f, ϕ_f) . By the definition of E_f , it implies that

$$x^p \cdot (c + d) = x^p \cdot c + x^p \cdot d + \sum_{i=0}^{p-1} x^i \cdot f(x)(x^{p-1-i} \cdot d)$$

for $x \in \mathfrak{g}_0$, $c \in K$ and $d \in N$. By K, N are restricted modules and Lemma 2.7 (1), $x^p \cdot (c + d) = x^{[p]} \cdot c + x^{[p]} \cdot d + D_{x^{p-1}}(f(x)) = x^{[p]} \cdot c + x^{[p]} \cdot d + x^{p-1} \cdot f(x) = x^{[p]} \cdot c + x^{[p]} \cdot d + f(x^{[p]}) = x^{[p]} \cdot (c + d)$. Hence, $V \subseteq Fi_2(\text{Ext}_*(K, N))$ and therefore $V = Fi_2(\text{Ext}_*(K, N))$.

Let f be Lie type 1-cocycle and assume its cohomology class belongs to $\text{Im } i_1$. Thus there is an associative type 1-cocycle $g \in \text{Hom}(\mathfrak{u}(\mathfrak{g})^+, \underline{\text{Hom}}_k(N, K))$ such that the cohomology class of g^0 is same as that of f . Thus there is an element $m \in \underline{\text{Hom}}_k(N, K)$ such that $f(x) = g(x) + x \cdot m$ for $x \in \mathfrak{g}$. Note that g is 1-cocycle, $g(xy) = x \cdot g(y)$ for any $x, y \in \mathfrak{u}(\mathfrak{g})^+$. Therefore, $x^{p-1} \cdot f(x) = x^{p-1} \cdot g(x) + x^p \cdot m = g(x^{[p]}) + x^{[p]} \cdot m = f(x^{[p]})$. So $\text{Im } i_1 \subseteq V$. Conversely, let $f \in \text{Hom}_k(\mathfrak{g}, \underline{\text{Hom}}_k(N, K))$ be any Lie type 1-cocycle satisfying $x^{p-1} \cdot f(x) = f(x^{[p]})$, for $x \in \mathfrak{g}_0$. Denote one of its corresponding associative type 1-cocycles by g . Thus there is an element $m \in \underline{\text{Hom}}_k(N, K)$ such that $g(x) = f(x) + x \cdot m$. So $g(x^p - x^{[p]}) = g(x^p) - f(x^{[p]}) - x^{[p]} \cdot m = x^{p-1} \cdot g(x) - f(x^{[p]}) - x^{[p]} \cdot m = x^{p-1} \cdot (f(x) + x \cdot m) - f(x^{[p]}) - x^{[p]} \cdot m = 0$. So g is indeed defined over $\mathfrak{u}(\mathfrak{g})$. Thus $V \subseteq \text{Im } i_1$. \square

4. EXTENSIONS OF RESTRICTED LIE SUPERALGEBRAS: THE SIMILARITY CLASSES

The definition of a Lie superalgebra extension has been described in Subsection 2.2. We also hope to consider the analogous case where Lie superalgebras are replaced by restricted ones. The definition of a restricted Lie superalgebra extension can be given directly. Let M be an abelian restricted Lie superalgebra and \mathfrak{g} just a restricted Lie superalgebra. A *restricted extension of M by \mathfrak{g}* is a pair (E, ϕ) where E is a restricted Lie superalgebra

containing M as an ideal, and ϕ is a restricted Lie superalgebra epimorphism $E \rightarrow \mathfrak{g}$ such that $\text{Ker } \phi = M$. Similarly, this situation defines on M the structure of a \mathfrak{g} -module and it is easy to see this module is restricted.

Obviously, there are two ways to consider the relations between different restricted extensions: *similarity classes* and *equivalence classes*. By definition, two restricted extensions (E, ϕ) and (E', ϕ') are said to be *similar* if there is a Lie superalgebra isomorphism $\alpha : E \rightarrow E'$ which leaves the elements of M fixed and satisfies the relation $\phi' \alpha = \phi$. And, they are *equivalent* if moreover α is a restricted map, that is, $\alpha(x^{[p]}) = \alpha(x)^{[p]}$ for $x \in E_{\bar{0}}$. In this section, we want to characterize the similarity classes by using cohomology theory. In subsection 2.2, we have used the notion $\text{Ext}(M, \mathfrak{g})$ to denote the set of ordinary equivalence classes. To not cause confusion, we introduce two more notions. The set of similarity classes and equivalence classes of restricted extensions of M by \mathfrak{g} are denoted by $\text{Ext}_0(M, \mathfrak{g})$ and $\text{Ext}_*(M, \mathfrak{g})$, respectively. They are abelian groups. Clearly, we have two natural maps of abelian groups

$$(4.1) \quad i_3 : \text{Ext}_0(M, \mathfrak{g}) \hookrightarrow \text{Ext}(M, \mathfrak{g}), \quad \pi_1 : \text{Ext}_*(M, \mathfrak{g}) \twoheadrightarrow \text{Ext}_0(M, \mathfrak{g})$$

where i_3 is injective and π_1 is surjective. As we will see later, both $\text{Ext}_0(M, \mathfrak{g})$ and $\text{Ext}_*(M, \mathfrak{g})$ are ordinary linear spaces whenever M is *strongly abelian*. And in such case, above two maps are linear maps automatically.

Definition 4.1. *A restricted Lie superalgebra M is strongly abelian if it is abelian and $x^{[p]} = 0$ for all $x \in M_{\bar{0}}$.*

Lemma 4.2. *Every restricted extension of M by \mathfrak{g} is similar to one in which M is strongly abelian.*

Proof. Note that we can generalize the Proposition 2.1 in [9] to the Lie superalgebra directly. That is, for a restricted Lie superalgebra $(\mathfrak{g}, [p])$ and a map $[p]_1 : \mathfrak{g} \rightarrow \mathfrak{g}$, $[p]_1$ is still a p -mapping of \mathfrak{g} if and only if $[p] - [p]_1$ is a p -semilinear map from \mathfrak{g} to $C(\mathfrak{g})$, where $C(\mathfrak{g})$ is the center of \mathfrak{g} .

Now let (E, ϕ) be a restricted extension. By the definition of a p -mapping and M is an abelian ideal of E , $M^{[p]}$ is contained in the center of E . So the restriction of $[p]$ to M sends M to $C(E)$ and this map is p -semilinear. We evidently extend this map to a p -semilinear map g from E to $C(E)$. Thus by the result stated in the above paragraph, $[p]_1 := [p] - g$ is another p -mapping of E . Equipping with this new p -mapping, we get a restricted extension of M by \mathfrak{g} which is clearly similar to the given extension by the identity map, and in which M is strongly abelian. \square

By this lemma, this is no harm to assume that M is strong abelian, and we indeed do so in the following of this section, when we only consider

similarity classes. By Lemma 2.5, there is an isomorphism between $\text{Ext}(M, \mathfrak{g})$ and $H^2(\mathfrak{g}, M)$. Since $\text{Ext}_0(M, \mathfrak{g})$ is a subset of $\text{Ext}(M, \mathfrak{g})$, there is a subset of $H^2(\mathfrak{g}, M)$ which corresponds to $\text{Ext}_0(M, \mathfrak{g})$. For convenience, denote this subset by $H_0^2(\mathfrak{g}, M)$. So our aim is to characterize $H_0^2(\mathfrak{g}, M)$ by using cohomologies.

Now, let f be a 2-cocycle. Recall from the proof of Lemma 2.5 the corresponding extension is (E_f, ϕ_f) . For $x \in \mathfrak{g}_0, x_1 \in \mathfrak{g}$, direct computations show that

$$D_{(x,0)^p}(x_1, 0) = ([x^{[p]}, x_1], \sum_{i=0}^{p-1} x^i \cdot f(x, D_{x^{p-1-i}}(x_1))).$$

For short, define $k_x(x_1) := \sum_{i=0}^{p-1} x^i \cdot f(x, D_{x^{p-1-i}}(x_1))$ and $f_{x_1}(x) := f(x, x_1)$.

Lemma 4.3. *The map $k_x + f_{x^{[p]}}$ for any $x \in \mathfrak{g}_0$ is a 1-cocycle from \mathfrak{g} to M and it only depends on the cohomology class of f .*

Proof. By $x \in \mathfrak{g}_0$, $D_{(x,0)^p}$ is an ordinary derivation, that is

$$D_{(x,0)^p}[(x_1, 0), (x_2, 0)] = [D_{(x,0)^p}(x_1, 0), (x_2, 0)] + [(x_1, 0), D_{(x,0)^p}(x_2, 0)]$$

for $x_1, x_2 \in \mathfrak{g}$. From this, we have

$$\begin{aligned} k_x([x_1, x_2]) + x^{[p]} \cdot f(x_1, x_2) &= x_1 \cdot k_x(x_2) - (-1)^{|x_1||x_2|} x_2 \cdot k_x(x_1) \\ &\quad + f(x_1, [x^{[p]}, x_2]) + f([x^{[p]}, x_1], x_2). \end{aligned}$$

Since f is a 2-cocycle, $x^{[p]} \cdot f(x_1, x_2) - f(x_1, [x^{[p]}, x_2]) - f([x^{[p]}, x_1], x_2) = -x_1 \cdot f(x_2, x^{[p]}) + (-1)^{|x_1||x_2|} x_2 \cdot f(x_1, x^{[p]}) + f([x_1, x_2], x^{[p]}) = -x_1 \cdot f_{x^{[p]}}(x_2) + (-1)^{|x_1||x_2|} x_2 \cdot f_{x^{[p]}}(x_1) + f_{x^{[p]}}([x_1, x_2])$. From this, it is easy to see that $k_x + f_{x^{[p]}}$ is indeed a 1-cocycle. To show the second claim, we need show $k_x + f_{x^{[p]}}$ is a coboundary whenever f is so. Now assume that $f = \delta g$ for some $g \in \text{Hom}_k(\mathfrak{g}, M)$. Then

$$\begin{aligned} k_x(x_1) &= \sum_{i=0}^{p-1} x^i \cdot f(x, D_{x^{p-1-i}}(x_1)) \\ &= \sum_{i=0}^{p-1} x^i \cdot (x \cdot g(D_{x^{p-1-i}}(x)) - D_{x^{p-1-i}}(x) \cdot g(x) - g(D_{x^{p-i}}(x))) \\ &= x^{[p]} \cdot g(x_1) - g([x^{[p]}, x_1]) - \sum_{i=0}^{p-1} x^i D_{x^{p-1-i}}(x_1) \cdot g(x) \\ &= x^{[p]} \cdot g(x_1) - g([x^{[p]}, x_1]) - \sum_{i=0}^{p-1} (-1)^i \binom{p}{i+1} x^{p-1-i} x_1 x^i \cdot g(x) \\ &= x^{[p]} \cdot g(x_1) - g([x^{[p]}, x_1]) - x_1 x^{p-1} \cdot g(x), \end{aligned}$$

where Lemma 2.7 (2) is used. Therefore, $k_x(x_1) + f_{x^{[p]}}(x_1) = x^{[p]} \cdot g(x_1) - g([x^{[p]}, x_1]) - x_1 x^{p-1} \cdot g(x) + x_1 \cdot (g(x^{[p]})) - x^{[p]} \cdot g(x_1) - g([x_1, x^{[p]}]) = x_1 \cdot (g(x^{[p]}) - x^{p-1} \cdot g(x))$. So $k_x + f_{x^{[p]}}$ is a coboundary too. \square

By this lemma, for any $x \in \mathfrak{g}_{\bar{0}}$, we get a map $\Phi_x : H^2(\mathfrak{g}, M) \rightarrow H^1(\mathfrak{g}, M)$ which is induced by $f \mapsto k_x + f_{x^{[p]}}$. The cohomology class of f is denoted by $c(f)$ and we want to give a more controllable representative to $\Phi_x(c(f))$. For this, let g be an associative type 2-cocycle whose cohomology class is $c(f)$. So we can assume

$$f(x_1, x_2) = g(x_1, x_2) - (-1)^{|x_1||x_2|} g(x_2, x_1)$$

for $x_1, x_2 \in \mathfrak{g}$. Using this associative type 2-cocycle and noting $x \in \mathfrak{g}_{\bar{0}}$,

$$\begin{aligned} k_x(x_1) &= \sum_{i=0}^{p-1} x^i \cdot f(x, D_{x^{p-1-i}}(x_1)) \\ &= \sum_{i=0}^{p-1} x^i \cdot (g(x, D_{x^{p-1-i}}(x_1)) - g(D_{x^{p-1-i}}(x_1), x)) \\ &= \sum_{i=0}^{p-1} g(x^{i+1}, D_{x^{p-1-i}}(x_1)) - \sum_{i=0}^{p-1} g(x^i, x D_{x^{p-1-i}}(x_1)) \\ &\quad - \sum_{i=0}^{p-1} g(x^i D_{x^{p-1-i}}(x_1), x) + \sum_{i=0}^{p-1} g(x^i, D_{x^{p-1-i}}(x_1) x) \\ &= \sum_{i=0}^{p-1} g(x^{i+1}, D_{x^{p-1-i}}(x_1)) - \sum_{i=0}^{p-1} g(x^i, D_{x^{p-i}}(x_1)) \\ &\quad - \sum_{i=0}^{p-1} g(x^i D_{x^{p-1-i}}(x_1), x) \\ &= g(x^p, x_1) - \sum_{i=0}^{p-1} g(x^i D_{x^{p-1-i}}(x_1), x) \\ &= g(x^p, x_1) - \sum_{i=0}^{p-1} g((-1)^i \binom{p}{i+1} x^{p-1-i} x_1 x^i, x) \\ &= g(x^p, x_1) - g(x_1 x^{p-1}, x) \\ &= g(x^p, x_1) - g(x_1, x^p) - x_1 \cdot g(x^{p-1}, x). \end{aligned}$$

Therefore,

$$k_x(x_1) + f_{x^{[p]}}(x_1) = g(x^p - x^{[p]}, x_1) - g(x_1, x^p - x^{[p]}) - x_1 \cdot g(x^{p-1}, x).$$

Hence, $\Phi_x(c(f))$ has a representative 1-cocycle $g'_x : x_1 \mapsto g(x^p - x^{[p]}, x_1) - g(x_1, x^p - x^{[p]})$. By this expression, we get a p -semilinear map $g' : \mathfrak{g}_{\bar{0}} \rightarrow$

$H^1(\mathfrak{g}, M)$ induced by $x \mapsto g'_x$. Thus we get a linear map

$$(4.2) \quad \Phi : H^2(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, H^1(\mathfrak{g}, M)), \quad g \mapsto g'.$$

The characterization of $H_0^2(\mathfrak{g}, M)$ is described as follows.

Proposition 4.4. $H_0^2(\mathfrak{g}, M) = \text{Ker } \Phi$.

Proof. Let f be a 2-cocycle and assume $c(f) \in H_0^2(\mathfrak{g}, M)$. Thus the corresponding extension (E_f, ϕ_f) is a restricted extension. So for any $x \in \mathfrak{g}_0$, there is an element $\rho(x) \in M$ such that $(x, 0)^{[p]} = (x^{[p]}, \rho(x))$. From $D_{(x,0)^p} = D_{(x,0)^{[p]}}$, we must have $k_x(x_1) = f(x^{[p]}, x_1) - x_1 \cdot \rho(x)$, i.e., $k_x + f_{x^{[p]}} = \delta(-\rho(x))$, so that $\Phi_x(c(f)) = 0$. Therefore $c(f) \in \text{Ker } \Phi$ and thus $H_0^2(\mathfrak{g}, M) \subseteq \text{Ker } \Phi$.

Conversely, assume $c(f) \in \text{Ker } \Phi$. Recall we use g to denote the associative type 2-cocycle of f such that $f(x_1, x_2) = g(x_1, x_2) - (-1)^{|x_1||x_2|}g(x_2, x_1)$. Thus we get a p -semilinear map

$$\sigma : \mathfrak{g}_0 \rightarrow M_0$$

such that $g'_x(x_1) = x_1 \cdot \sigma(x)$. Now we define a p -mapping on E_f through

$$(4.3) \quad (x, m)^{[p]} := (x^{[p]}, x^{p-1} \cdot m + g(x^{p-1}, x) - \sigma(x))$$

for $x \in \mathfrak{g}_0, m \in M_0$. Of course, one can show directly (4.3) indeed gives a p -mapping on E_f and thus E_f is a restricted Lie superalgebra. Also, one can copy the same computations used in restricted Lie algebra case (see p. 568-569 in [5]) to show (4.3) satisfy all conditions of a p -mapping. In one word, $c(f) \in H_0^2(\mathfrak{g}, M)$ and thus $\text{Ker } \Phi \subseteq H_0^2(\mathfrak{g}, M)$. \square

Remark 4.5. By this proposition, if M is strongly abelian, we know that $\text{Ext}_0(M, \mathfrak{g})$ is also an ordinary vector space and the canonical map i_3 given in (4.1) is a linear map.

5. EXTENSIONS OF RESTRICTED LIE SUPERALGEBRAS: THE EQUIVALENCE CLASSES

Further, we consider the restricted equivalence classes $\text{Ext}_*(M, \mathfrak{g})$ in this section. And, as the final conclusion, the Hochschild's 6-term exact sequence will be given. As the beginning, a decomposition of a similarity class into equivalence classes will be given.

5.1. Decomposition of similarity classes. Let (E, ϕ) be a restricted extension of M by \mathfrak{g} and denote its similarity class by c . We want to decompose c into a set S_c of equivalence classes. For any other representative object (E', ϕ') of c , there is a similarity isomorphism $\gamma : (E, \phi) \rightarrow (E', \phi')$.

Lemma 5.1. *For any $e \in E_0$, $\gamma(e^{[p]}) - (\gamma(e))^{[p]}$ depends only on $\phi(e)$.*

Proof. To attack it, it is enough to show that $\gamma(e^{[p]}) - (\gamma(e))^{[p]} = \gamma((e + m)^{[p]}) - (\gamma(e + m))^{[p]}$ for any $m \in M_{\bar{0}}$. This is just a direct computation.

$$\gamma((e + m)^{[p]}) = \gamma(e^{[p]} + m^{[p]} + e^{p-1} \cdot m) = \gamma(e^{[p]}) + m^{[p]} + e^{p-1} \cdot m,$$

$$(\gamma(e + m))^{[p]} = (\gamma(e) + m)^{[p]} = \gamma(e)^{[p]} + m^{[p]} + \gamma(e)^{p-1} \cdot m.$$

Note that we always have $e^{p-1} \cdot m = \gamma(e)^{p-1} \cdot m$, we get $\gamma(e^{[p]}) - (\gamma(e))^{[p]} = \gamma((e + m)^{[p]}) - (\gamma(e + m))^{[p]}$. \square

By this lemma, for any $e \in E_{\bar{0}}$, one can denote the difference $\gamma(e^{[p]}) - (\gamma(e))^{[p]}$ by $g(\phi(e))$ and hence we get a map

$$g : \mathfrak{g}_{\bar{0}} \rightarrow M_{\bar{0}}.$$

Lemma 5.2. *g is a p -semilinear map from $\mathfrak{g}_{\bar{0}}$ to $M_{\bar{0}}^{\mathfrak{g}}$ where $M_{\bar{0}}^{\mathfrak{g}} := \{m \in M_{\bar{0}} | x \cdot m = 0, x \in \mathfrak{g}\}$.*

Proof. By γ is a Lie superalgebra map,

$$\gamma\left(\sum_{i=0}^{p-1} s_i(e_1, e_2)\right) = \sum_{i=0}^{p-1} s_i(\gamma(e_1), \gamma(e_2))$$

for $e_1, e_2 \in E_{\bar{0}}$ (See condition (c) of a p -mapping for the definition of $s_i(x, y)$). This indeed implies that g is a p -semilinear map.

Furthermore, for any $x = \phi(z) \in \mathfrak{g}$, we have

$$\begin{aligned} x \cdot g(\phi(e)) &= \gamma(z) \cdot (\gamma(e^{[p]}) - (\gamma(e))^{[p]}) \\ &= \gamma([z, e^{[p]}]) + D_{\gamma(e)^p}(\gamma(z)) \\ &= \gamma([z, e^{[p]}]) + \gamma(D_{e^p}(z)) \\ &= 0. \end{aligned}$$

\square

As stated in the first paragraph of the proof of Lemma 4.2, one can give a new p -mapping for E by setting $e^{(p)} := e^{[p]} - g(\phi(e))$ and thus we get a new restricted Lie superalgebra which gives an extension of M by \mathfrak{g} . Denote it by (E_g, ϕ) . Now $\gamma(e^{(p)}) = \gamma(e^{[p]}) - g(\phi(e)) = \gamma(e)^{[p]}$ and so (E_g, ϕ) is equivalent to (E', ϕ') . Conversely, for any $g \in S(\mathfrak{g}_{\bar{0}}, M_{\bar{0}}^{\mathfrak{g}})$, (E_g, ϕ) is similar to (E, ϕ) . Moreover, we get an action

$$g^* : \text{Ext}_*(M, \mathfrak{g}) \rightarrow \text{Ext}_*(M, \mathfrak{g}), \quad (E, \phi) \mapsto (E_g, \phi).$$

Such discussions give the following basic fact.

Lemma 5.3. *Through the map $g \mapsto g^*$, $S(\mathfrak{g}_{\bar{0}}, M_{\bar{0}}^{\mathfrak{g}})$ operates transitively on each S_c .*

We hope to determine the kernel of the map $g \mapsto g^*$. For this, we need give a cohomology explanation to the automorphisms of ordinary extensions. Let (F, ψ) be an ordinary extension of M by \mathfrak{g} , where "ordinary" means the extension need not to be restricted. An automorphism of (F, ψ) is an isomorphism of Lie superalgebras $\alpha : F \rightarrow F$ which leaves the elements of M fixed and satisfies that relation $\psi\alpha = \psi$. Since $\alpha(e) - e = \alpha(e + m) - (e + m)$ for any $m \in M$, $\alpha(e) - e$ only depends on $\psi(e)$ and denote it by $h(\psi(e))$. From this we get an even linear map $h : \mathfrak{g} \rightarrow M$, $\psi(e) \mapsto h(\psi(e))$ for $e \in F$.

Lemma 5.4. *The $h \in \text{Hom}(\mathfrak{g}, M)$ defined above is a 1-cocycle.*

Proof. For any $e_1, e_2 \in F$, we always have $\alpha([e_1, e_2]) = [\alpha(e_1), \alpha(e_2)]$. From this, we have $h([\psi(e_1), \psi(e_2)]) = [e_1, h(\psi(e_2))] - (-1)^{|e_1||e_2|}[e_2, h(\psi(e_1))] = [\psi(e_1), h(\psi(e_2))] - (-1)^{|e_1||e_2|}[\psi(e_2), h(\psi(e_1))]$. This implies that h is a 1-cocycle. \square

Conversely, for any 1-cocycle h one can get an automorphism of (F, ψ) by setting $\alpha : F \rightarrow F$, $e \mapsto e + h(\psi(e))$.

Now we go back to determine the kernel of the map $g \mapsto g^*$. Assume $g \in S(\mathfrak{g}_0, M_0^{\mathfrak{g}})$ is one lying in the kernel. So (E_g, ϕ) is restricted equivalent to (E, ϕ) for any restricted extension (E, ϕ) . Let $\gamma : E \rightarrow E_g$ be the isomorphism. By forgetting the restricted structure, γ gives an automorphism of (E, ϕ) . Owing to Lemma 5.4, $\gamma(e) = e + h(\phi(e))$ for some 1-cocycle h . Then by $\gamma(e^{[p]}) = \gamma(e)^{[p]}$ for $e \in E_0$, $g(\phi(e)) = e^{p-1} \cdot h(\phi(e)) + h(\phi(e))^{[p]} - h(\phi(e)^{[p]})$. That is, for any $x \in \mathfrak{g}_0$, we have

$$g(x) = x^{p-1} \cdot h(x) + h(x)^{[p]} - h(x^{[p]}).$$

Define $h'(x) := x^{p-1} \cdot h(x) + h(x)^{[p]} - h(x^{[p]})$ and it is not hard to see that $h' \in S(\mathfrak{g}_0, M_0^{\mathfrak{g}})$. So we get a liner map

$$(5.1) \quad \Psi : Z^1(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, M_0^{\mathfrak{g}}), \quad h \mapsto h',$$

where as usual $Z^1(\mathfrak{g}, M)$ is the space of 1-cocycles for \mathfrak{g} in M . Now, we know that $\text{Im } \Psi$ is just the kernel of the map $g \mapsto g^*$. Note that the procedure to determine the kernel does not depends on the choice of equivalence class of (E, ϕ) . So we can choose it to be the trivial extension s_0 , that is, the 0-element in $\text{Ext}_*(M, \mathfrak{g})$. Define

$$G_{s_0} : S(\mathfrak{g}_0, M_0^{\mathfrak{g}}) \rightarrow \text{Ext}_*(M, \mathfrak{g}), \quad g \mapsto g^*(s_0).$$

So $\text{Im } \Psi = \text{Ker } G_{s_0}$. Thus, the following 4-term exact sequence is gotten.

Proposition 5.5. *Let M be an abelian restricted Lie superalgebra on which the restricted Lie superalgebra \mathfrak{g} operates. Then we have the following 4-term*

exact sequence of abelian groups

$$Z^1(\mathfrak{g}, M) \xrightarrow{\Psi} S(\mathfrak{g}_0, M_0^{\mathfrak{g}}) \xrightarrow{G_{s_0}} \text{Ext}_*(M, \mathfrak{g}) \xrightarrow{\pi_1} \text{Ext}_0(M, \mathfrak{g}) \rightarrow 0.$$

5.2. Equivalence classes. In this subsection, we always assume that M is strongly abelian. In such case, the connection between $\text{Ext}_*(M, \mathfrak{g})$ and $H_*^2(\mathfrak{g}, M)$ is nice.

Proposition 5.6. *Let M be a strongly abelian restricted Lie superalgebra on which the restricted Lie superalgebra \mathfrak{g} operates. Then there is a canonical isomorphism $F|_* : \text{Ext}_*(M, \mathfrak{g}) \xrightarrow{\cong} H_*^2(\mathfrak{g}, M)$ such that the following diagram of canonical maps is commutative*

$$\begin{array}{ccc} \text{Ext}_*(M, \mathfrak{g}) & \xrightarrow{F|_*} & H_*^2(\mathfrak{g}, M) \\ \downarrow \pi_1 & & \downarrow \\ \text{Ext}_0(M, \mathfrak{g}) & \xrightarrow{F|_0} & H_0^2(\mathfrak{g}, M) \end{array}$$

where $F|_0$ is the restriction of the canonical isomorphism $F : \text{Ext}(M, \mathfrak{g}) \xrightarrow{\cong} H^2(\mathfrak{g}, M)$ to $\text{Ext}_0(M, \mathfrak{g})$.

Proof. The idea is similar to that used in the proof of Lemma 2.5. Given a restricted associative type 2-cocycle g of $\mathfrak{u}(\mathfrak{g})$ with values in M . Construct the corresponding restricted extension (E_g, ϕ_g) as follows: As a space, $E_g = \mathfrak{g} \oplus M$, $\phi_g(x_1, m_1) = x_1$ and the restricted Lie superalgebra structure is given through

$$\begin{aligned} [(x_1, m_1), (x_2, m_2)] &:= ([x_1, x_2], x_1 \cdot m_1 - (-1)^{|x_1||x_2|} x_2 \cdot m_1 \\ &\quad + g(x_1, x_2) - (-1)^{|x_1||x_2|} g(x_2, x_1)), \\ (x, m)^{[p]} &:= (x^{[p]}, x^{p-1} \cdot m + g(x^{p-1}, x)) \end{aligned}$$

for $x_1, x_2 \in \mathfrak{g}, x \in \mathfrak{g}_0$ and $m_1, m_2 \in M, m \in M_0$. It is straightforward to show (E_g, ϕ_g) is indeed a restricted extension of M by \mathfrak{g} . Next, we need to show that the equivalence class of (E_g, ϕ_g) depends only on the cohomology class in $H_*^2(\mathfrak{g}, M)$ of g . Let h be a 1-cochain and we will show that $(E_{g+\delta(h)}, \phi_{g+\delta(h)})$ is equivalent to (E_g, ϕ_g) . In fact, define

$$\alpha : E_{g+\delta(h)} \rightarrow E_g, \quad (x, m) \mapsto (x, m + h(x))$$

and direct computations show that α is an equivalence isomorphism. Thus the map $g \mapsto (E_g, \phi_g)$ induces a linear homomorphism

$$G|_* : H_*^2(\mathfrak{g}, M) \rightarrow \text{Ext}_*(M, \mathfrak{g}).$$

Conversely, let (E, ϕ) be a restricted extension of M by \mathfrak{g} and ϕ can be extended uniquely to a homomorphism ϕ' from $\mathfrak{u}(E)$ to $\mathfrak{u}(\mathfrak{g})$. Clearly,

$\text{Ker}(\phi') = \mathbf{u}(E)M$. Note that we can not apply the same method used in the proof of Lemma 2.5 directly since it only gives us a 2-cocycle of Lie type. And in the restricted case, we only have associative type cochains. To overcome this difficulty, a linear map is needed. It is known that M is a $\mathbf{u}(E)$ -module. We claim the identity map of M to M can be extended in one and only way to a $\mathbf{u}(E)$ -homomorphism from $\mathbf{u}(E)M$ to M , each being regarded as a $\mathbf{u}(E)$ -module in the natural fashion. Actually, the map

$$\gamma : \mathbf{u}(E)M \rightarrow M, \quad \sum_i u_i m_i \mapsto \sum_i \phi'(u_i) \cdot m_i$$

is the desired one. Now, we go back to construct an associative type 2-cocycle. To attack it, let ψ be a linear map from \mathfrak{g} to E which is inverse to ϕ . We can extend ψ to a linear map $\psi' : \mathbf{u}(\mathfrak{g}) \rightarrow \mathbf{u}(E)$ such that $\phi'\psi' = \text{id}_{\mathbf{u}(\mathfrak{g})}$. Now define

$$g : \mathbf{u}(\mathfrak{g})^+ \otimes \mathbf{u}(\mathfrak{g})^+ \rightarrow M, \quad (x, y) \mapsto \gamma(\psi'(x)\psi'(y) - \psi'(xy)).$$

One can check directly that g is an associative type 2-cocycle of $\mathbf{u}(\mathfrak{g})$ in M and this induces a linear map

$$F|_* : \text{Ext}_*(M, \mathfrak{g}) \rightarrow H_*^2(\mathfrak{g}, M).$$

Once we can show that (E, ϕ) is equivalent to (E_g, ϕ_g) , then $G|_* \circ F|_* = \text{id}_{\text{Ext}_*(M, \mathfrak{g})}$. Actually, define

$$\alpha : E_g \rightarrow E, \quad (x, m) \mapsto \psi(x) + m.$$

Through direct computations, we get

$$\begin{aligned} \alpha([(x_1, m_1), (x_2, m_2)]) &= \alpha([x_1, x_2], x_1 \cdot m_2 - (-1)^{|x_1||x_2|} x_2 \cdot m_1 \\ &\quad + g(x_1, x_2) - (-1)^{|x_1||x_2|} g(x_2, x_1)) \\ &= \psi([x_1, x_2]) + x_1 \cdot m_2 - (-1)^{|x_1||x_2|} x_2 \cdot m_1 \\ &\quad + \gamma([\psi(x_1), \psi(x_2)] - \psi([x_1, x_2])) \\ &= x_1 \cdot m_2 - (-1)^{|x_1||x_2|} x_2 \cdot m_1 + [\psi(x_1), \psi(x_2)] \\ &= [\alpha(x_1, m_1), \alpha(x_2, m_2)]. \end{aligned}$$

Furthermore,

$$\begin{aligned} \alpha(x, m)^{[p]} &= \psi(x)^{[p]} + x^{p-1} \cdot m, \\ \alpha((x, m)^{[p]}) &= \psi(x^{[p]}) + x^{p-1} \cdot m + g(x^{p-1}, x). \end{aligned}$$

And, $g(x^{p-1}, x) = \gamma(\psi'(x^{p-1})\psi(x) - \psi'(x^p)) = \gamma(\psi(x)^p - \psi(x^{[p]})) = \psi(x)^{[p]} - \psi(x^{[p]})$. So $\alpha(x, m)^{[p]} = \alpha((x, m)^{[p]})$ and hence α is an equivalence isomorphism.

At last, let us show that $G|_*$ is a monomorphism. Suppose that (E_g, ϕ_g) is a trivial extension. By definition, there is a homomorphism $\psi : \mathfrak{g} \rightarrow E_g$ of

restricted Lie superalgebras such that $\phi\psi = \text{id}_{\mathfrak{g}}$. Write $\psi(x_1) = (x_1, -h(x_1))$ for $x_1 \in \mathfrak{g}$. By ψ is a homomorphism of restricted Lie superalgebras, we get

$$(5.2) \quad g(x_1, x_2) - (-1)^{|x_1||x_2|}g(x_2, x_1) = x_1 \cdot h(x_2) - (-1)^{|x_1||x_2|}x_2 \cdot h(x_1) - h([x_1, x_2]),$$

$$(5.3) \quad g(x^{p-1}, x) = x^{p-1} \cdot h(x) - h(x^{[p]})$$

for $x_1, x_2 \in \mathfrak{g}$ and $x \in \mathfrak{g}_0$. The equation (5.2) implies that Lie type cocycle corresponding g is a coboundary. So g^0 is also a coboundary (see subsection 2.2 for the definition of g^0). That is, there is a 1-cochain ω for $U(\mathfrak{g})^+$ in M such that $g^0(u, v) = u \cdot \omega(v) - \omega(uv)$ for $u, v \in U(\mathfrak{g})^+$. So it is not hard to see $\omega|_{\mathfrak{g}} - h$ is a Lie type 1-cocycle and therefore coincides $\varpi|_{\mathfrak{g}}$ for an associative type 1-cocycle ϖ for $U(\mathfrak{g})^+$ in M . Replacing ω by $\omega - \varpi$, one can assume the $\omega|_{\mathfrak{g}} = h$. So (5.3) implies that $g(x^{p-1}, x) = x^{p-1} \cdot \omega(x) - \omega(x^{[p]})$. At the same time, $g(x^{p-1}, x) = g^0(x^{p-1}, x) = x^{p-1} \cdot \omega(x) - \omega(x^p)$. Thus $\omega(x^{[p]}) = \omega(x^p)$ and so $\omega = f^0$ with f a 1-cochain for $\mathfrak{u}(\mathfrak{g})^+$ in M . Therefore, $g = \delta f$ and we get the desired conclusion. \square

5.3. The 6-term exact sequence. Now the Hochschild's 6-term exact sequence is a direct consequence of conclusions we built.

Theorem 5.7. *Let M be a strongly abelian restricted Lie superalgebra on which the restricted Lie superalgebra \mathfrak{g} operates. Then we have the following 6-term exact sequence of ordinary linear spaces*

$$0 \rightarrow H_*^1(\mathfrak{g}, M) \xrightarrow{i_1} H^1(\mathfrak{g}, M) \xrightarrow{\bar{\Psi}} S(\mathfrak{g}_0, M_0^{\mathfrak{g}}) \\ \xrightarrow{F|_* \circ G_{s_0}} H_*^2(\mathfrak{g}, M) \xrightarrow{F|_0 \circ \pi_1 \circ F|_*^{-1}} H^2(\mathfrak{g}, M) \xrightarrow{\Phi} S(\mathfrak{g}_0, H^1(\mathfrak{g}, M)).$$

Proof. Recall the definition of $\Psi : Z^1(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, M_0^{\mathfrak{g}})$, one can find Ψ maps each 1-coboundary to 0 whenever M is strongly abelian. So Ψ induces a linear map $\bar{\Psi} : H^1(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, M_0^{\mathfrak{g}})$ naturally. By combining the description of $H_*^1(\mathfrak{g}, M)$ given in the proof of Proposition 3.2, we have the following exact sequence

$$0 \rightarrow H_*^1(\mathfrak{g}, M) \xrightarrow{i_1} H^1(\mathfrak{g}, M) \xrightarrow{\bar{\Psi}} S(\mathfrak{g}_0, M_0^{\mathfrak{g}}).$$

Through using Proposition 5.5, we get the exact sequence

$$0 \rightarrow H_*^1(\mathfrak{g}, M) \rightarrow H^1(\mathfrak{g}, M) \rightarrow S(\mathfrak{g}_0, M_0^{\mathfrak{g}}) \rightarrow \text{Ext}_*(M, \mathfrak{g}) \rightarrow \text{Ext}_0(M, \mathfrak{g}) \rightarrow 0.$$

By combining the descriptions of $\text{Ext}_0(M, \mathfrak{g})$ and $\text{Ext}_*(M, \mathfrak{g})$ given in Propositions 4.4 and 5.6 respectively, the desired 6-term exact sequence is followed. \square

Remark 5.8. (1) In page 575 in [5], Hochschild gave the 6-term exact sequence in the following way

$$\begin{aligned} 0 \rightarrow H_*^1(L, M) &\rightarrow H^1(L, M) \rightarrow S(L, M^L) \\ &\rightarrow H_*^2(L, M) \rightarrow H^2(L, M) \rightarrow S(L, H^1(L, M)), \end{aligned}$$

where L is a restricted Lie algebra and M is a strongly abelian restricted Lie algebra with an L -operation. Clearly, if we take \mathfrak{g} in Theorem 5.7 to be a restricted Lie algebra, then we recover the original Hochschild's 6-term exact sequence very well.

(2) As we have seen, the proof of main result depends on the interpretations of cohomology groups by using various kinds of extensions. It is hopeful that one can get the same result or find applications by filtering the associative cochain complex for $U(\mathfrak{g})$ in M relative the ideal, which generated by $x^p - x^{[p]}$ for $x \in \mathfrak{g}_0$, and considering the corresponding spectral sequence. This procedure should relate the works in [2, 3] to super case.

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DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, CHINA
E-mail address: `gxliu@nju.edu.cn`